Uncertainties in model-independent extractions of amplitudes from complete experiments

Sam Hoblit

Brookhaven National Lab

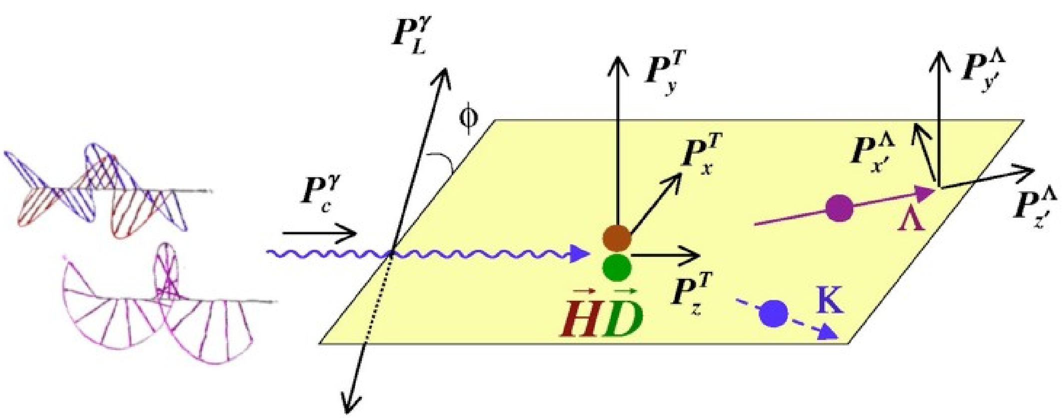
Extraction of Amplitudes

- Channels in the resonance region with possibility for complete experiments: πN, KΛ
- Avoiding ambiguities will require asymmetries involving recoil polarization.

  $\gamma p \rightarrow K^+\Lambda$, $\gamma n \rightarrow K^0\Lambda$: use self-analyzing weak decays
- Collect data on all possible observables in ~4π detectors.
- Express observables in terms of amplitudes
- Fit amplitudes to data

Polarization observables for $J^{\pi} = 0^{-}$ meson photo-production

Photon beam		Target			Recoil			Target - Recoil								
					x'	y'	z'	x'	x'	x'	y'	y'	y'	z'	z'	z'
		x	y	z				x	y	z	x	y	z	x	y	z
unpolarized	σ_0		T			P		$T_{x'}$		$L_{x'}$		Σ		$T_{z'}$		$L_{z'}$
$P_L^{\gamma}\ \sin(2\phi_{\gamma})$		H		G	$O_{x'}$		$O_{z'}$		$C_{z'}$		E		F		$-C_{x'}$	
$P_L^{\gamma}\ \cos(2\phi_{\gamma})$	$-\Sigma$		$-P$			$-T$		$-L_{z'}$		$T_{z'}$		$-\sigma_0$		$L_{x'}$		$-T_{x'}$
circular P_c^{γ}		F		$-E$	$C_{x'}$		$C_{z'}$		$-O_{z'}$		G		$-H$		$O_{x'}$	



***16 different observables, each appearing twice*:**

- ***single-pol observables can be measured from double-pol asy***
- ***double-pol observables can be measured from triple-pol asy***

Observables from CGLN F_i

$$\sigma_0 = \left\{ |F_1|^2 + |F_2|^2 + \tfrac{1}{2}\sin^2\theta \cdot \left(|F_3|^2 + |F_4|^2 \right) + \Re e\left[\sin^2\theta \cdot \left(F_2^* F_3 + F_1^* F_4 + \cos\theta \cdot F_3^* F_4 \right) - 2\cos\theta \cdot F_1^* F_2 \right] \right\} \cdot \rho$$

$$\hat{\Sigma} = -\left[\tfrac{1}{2}\sin^2\theta \cdot \left\{ |F_3|^2 + |F_4|^2 \right\} + \sin^2\theta \cdot \Re e\left\{ F_2^* F_3 + F_1^* F_4 + \cos\theta \cdot \left(F_3^* F_4 \right) \right\} \right] \cdot \rho$$

$$\hat{T} = \Im m\left\{ \sin\theta \left[F_1^* F_3 - F_2^* F_4 + \cos\theta \cdot \left(F_1^* F_4 - F_2^* F_3 \right) - \sin^2\theta \cdot F_3^* F_4 \right] \right\} \cdot \rho$$

$$\hat{P} = \Im m\left\{ \sin\theta \left[-2F_1^* F_2 - F_1^* F_3 + F_2^* F_4 + \cos\theta \cdot \left(F_2^* F_3 - F_1^* F_4 \right) + \sin^2\theta \cdot F_3^* F_4 \right] \right\} \cdot \rho$$

$$\hat{E} = -\left[-|F_1|^2 - |F_2|^2 + \Re e\left\{ 2\cos\theta \cdot \left(F_1^* F_2 \right) - \sin^2\theta \cdot \left(F_2^* F_3 + F_1^* F_4 \right) \right\} \right] \cdot \rho$$

$$\hat{G} = +\sin^2\theta \cdot \Im m\left\{ F_2^* F_3 + F_1^* F_4 \right\} \cdot \rho$$

$$\hat{F} = \sin\theta \cdot \Re e\left[F_1^* F_3 - F_2^* F_4 - \cos\theta \cdot \left(F_2^* F_3 - F_1^* F_4 \right) \right] \cdot \rho$$

$$\hat{H} = -\sin\theta \cdot \Im m\left[2F_1^* F_2 + F_1^* F_3 - F_2^* F_4 + \cos\theta \cdot \left(F_1^* F_4 - F_2^* F_3 \right) \right] \cdot \rho$$

Observables ⇔ amplitudes, CGLN F_i

$$\hat{O}_{x'} = -\sin\theta \cdot \Im m\left[F_2^* F_3 - F_1^* F_4 + \cos\theta \cdot \left(F_2^* F_4 - F_1^* F_3 \right) \right] \cdot \rho$$

$$\hat{O}_{z'} = +\sin^2\theta \cdot \Im m\left[F_1^* F_3 + F_2^* F_4 \right] \cdot \rho$$

$$\hat{C}_{x'} = +\sin\theta \cdot \Re e\left\{ -|F_1|^2 + |F_2|^2 + F_2^* F_3 - F_1^* F_4 + \cos\theta \cdot \left(F_2^* F_4 - F_1^* F_3 \right) \right\} \cdot \rho$$

$$\hat{C}_{z'} = +\Re e\left\{ -2F_1^* F_2 + \cos\theta \left(|F_1|^2 + |F_2|^2 \right) - \sin^2\theta \cdot \left(F_1^* F_3 + F_2^* F_4 \right) \right\} \cdot \rho$$

J. Phys. G: Nucl. Part. Phys. **38** (2011) 053001

$$\hat{L}_{x'} = +\Re e\left\{ \sin\theta \left[|F_1|^2 - |F_2|^2 + \tfrac{1}{2}\sin^2\theta \cdot \left(|F_4|^2 - |F_3|^2 \right) - F_2^* F_3 + F_1^* F_4 + \cos\theta \left(F_1^* F_3 - F_2^* F_4 \right) \right] \right\} \cdot \rho$$

$$\hat{T}_{z'} = \Re e\left\{ \sin\theta \left[-F_2^* F_3 + F_1^* F_4 + \cos\theta \left(F_1^* F_3 - F_2^* F_4 \right) + \tfrac{1}{2}\sin^2\theta \cdot \left(|F_4|^2 - |F_3|^2 \right) \right] \right\} \cdot \rho$$

$$\hat{L}_{z'} = \Re e\left\{ 2F_1^* F_2 - \cos\theta \left(|F_1|^2 + |F_2|^2 \right) + \sin^2\theta \cdot \left(F_1^* F_3 + F_2^* F_4 + F_3^* F_4 \right) + \tfrac{1}{2}\cos\theta \sin^2\theta \cdot \left(|F_3|^2 + |F_4|^2 \right) \right\} \cdot \rho$$

$$\hat{T}_{x'} = \Re e\left\{ \sin^2\theta \left[-F_1^* F_3 - F_2^* F_4 - F_3^* F_4 - \tfrac{1}{2}\cos\theta \cdot \left(|F_3|^2 + |F_4|^2 \right) \right] \right\} \cdot \rho$$

Multipole analysis of $\gamma p \rightarrow K^+\Lambda$

- **published data:**

Data group	*Experiment*	*Observable*	*Eγ range / W range*	*ΔEγ / ΔW binning*	*Systematic Scale error*
1	CLAS-g11a	$d\sigma$	938 – 3814 1625 – 2835	10	±8% (*Eγ dependent*)
2	CLAS-g11a	P	938 – 3814 1625 – 2835	10	±0.05
⇛ 3	CLAS-g1c	$C_{x'}$, $C_{z'}$	1032 – 2741 1679 – 2454	**101**	±0.03 ⇚
4	CLAS-g1c	$d\sigma$	944 – 2950 1628 – 2533	25	±8% (*Eγ dependent*)
5	GRAAL	$O_{x'}$, $O_{z'}$	980 – 1466 1649 – 1906	50	±4%
6	GRAAL	P	980 – 1466 1649 – 1906	50	±3%
7	GRAAL	Σ	980 – 1466 1649 – 1906	50	±2%
8	GRAAL	T	980 – 1466 1649 – 1906	50	±5%
9	LEPS	Σ	1550 – 2350 1947 – 2300	100	±3%

- **single-energy analyses limited by observables with the coarsest granularity**

Use Fierz identities to impose consistency across data sets

- use Fierz relations to construct expressions with expectation = 0

$$\mathrm{F}_{\mathrm{L.BR}} = \Sigma P - C_{x'} O_{z'} + C_{z'} O_{x'} - T$$

$$\mathrm{F}_{\mathrm{S.br}} = \mathrm{O}_{x'}^2 + \mathrm{O}_{z'}^2 + \mathrm{C}_{x'}^2 + \mathrm{C}_{z'}^2 + \Sigma^2 - T^2 + P^2 - 1$$

- combine data sets and minimize:

$$\chi^2 = \sum_{E_\gamma} \sum_{\theta_K} \left\{ \left(\left[\frac{F_{L.BR}\left(f_i x_{i\theta}^{exp}\right)}{\delta F_{L.BR}\left(f_i \sigma_{x_{i\theta}}\right)} \right]^2_{i=2,3,.5-8} + \left[\frac{F_{S.br}\left(f_i x_{i\theta}^{exp}\right)}{\delta F_{S.br}\left(f_i \sigma_{x_{i\theta}}\right)} \right]^2_{i=2,3,.5-8} \right) \right\} + \sum_i \left(\frac{f_i - 1}{\sigma_{f_i}} \right)^2$$

scale factor for the i^{th} data set

systematic error of the i^{th} data set

Fitted scales for $K^+\Lambda$ asymmetries

Data group	*Experiment*	*Observable*	*fitted scale* (f_i)	*err*[f_i]
2	CLAS-g11a	P	1.000	0.049
3	CLAS-g1c	$C_{x'}$, $C_{z'}$	0.984	0.025
5	GRAAL	$O_{x'}$, $O_{z'}$	0.997	0.035
6	GRAAL	P	1.001	0.030
7	GRAAL	Σ	1.001	0.020
8	GRAAL	T	0.992	0.040

When data from all 16 observables becomes available, the Fierz scaling fits will contain 37 constraints

Multipole fitting procedure

- Fit asymmetry scales using Feirz identities
- Vary multipoles and cross section scales minimizing:

$$\chi^2 = \sum_{i=1}^{N_s} \left\{ \sum_{j=1}^{N_i} \left(\frac{f_i x_{ij}^{exp} - x_{ij}^{fit}(\vec{\zeta})}{f_i \sigma_{x_{ij}}} \right)^2 \right\} + \sum_{i=1,4} + \left(\frac{f_i - 1}{\sigma_{f_i}} \right)^2$$

scale factor for i^{th} data set,
vary for cross sections,
fix from I for asymmetries

systematic error on cross sections of the i^{th} data set

Multipole fitting

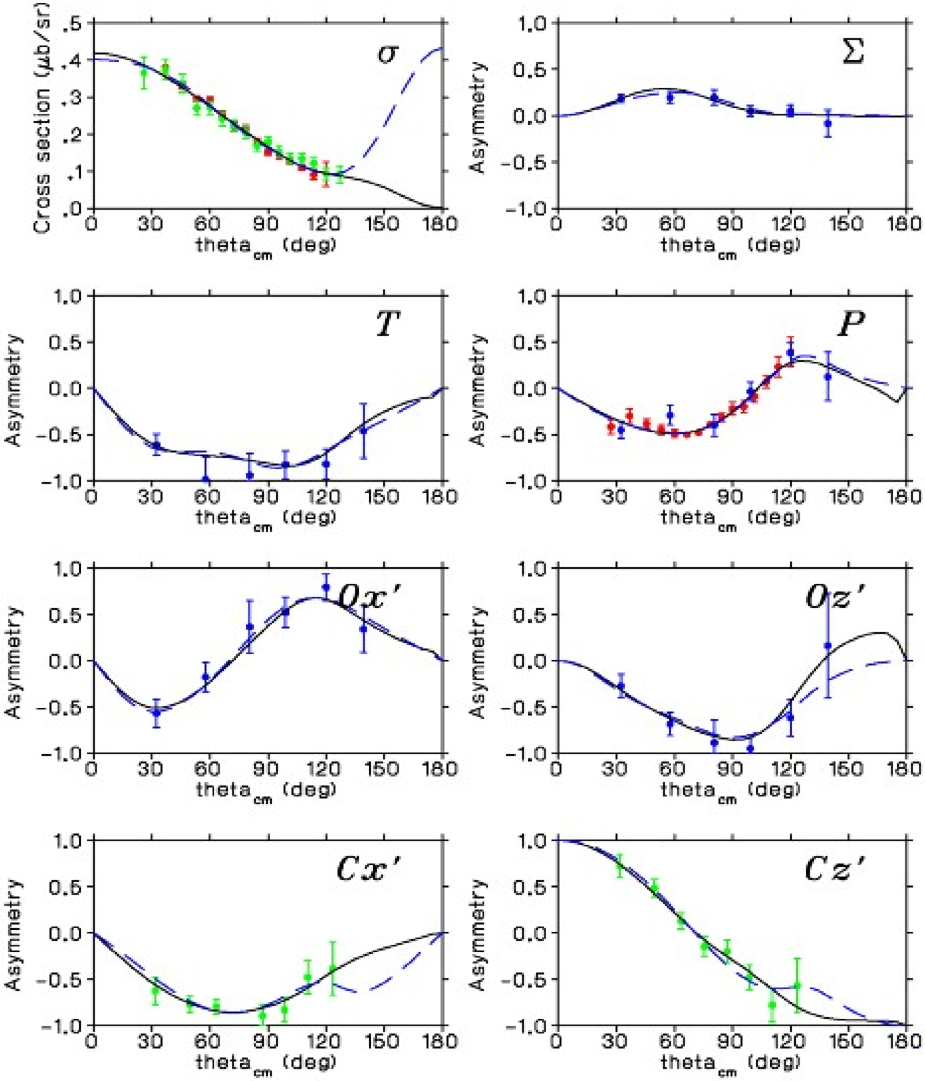
- Vary multipoles up to L=3
- Set L=4-8 to (real) Born values
- Monte-Carlo sample real & imaginary part of multipoles using generous sampling window
- Attempt gradient minimization using Minuit whenever random sampled χ^2 within a factor of 10^4 of current best
- Choose a reference point for overall phase, eg. $\mathrm{Im}(E_{0+}) = 0$

Results for combined g1c, g11a, and GRAAL data sets

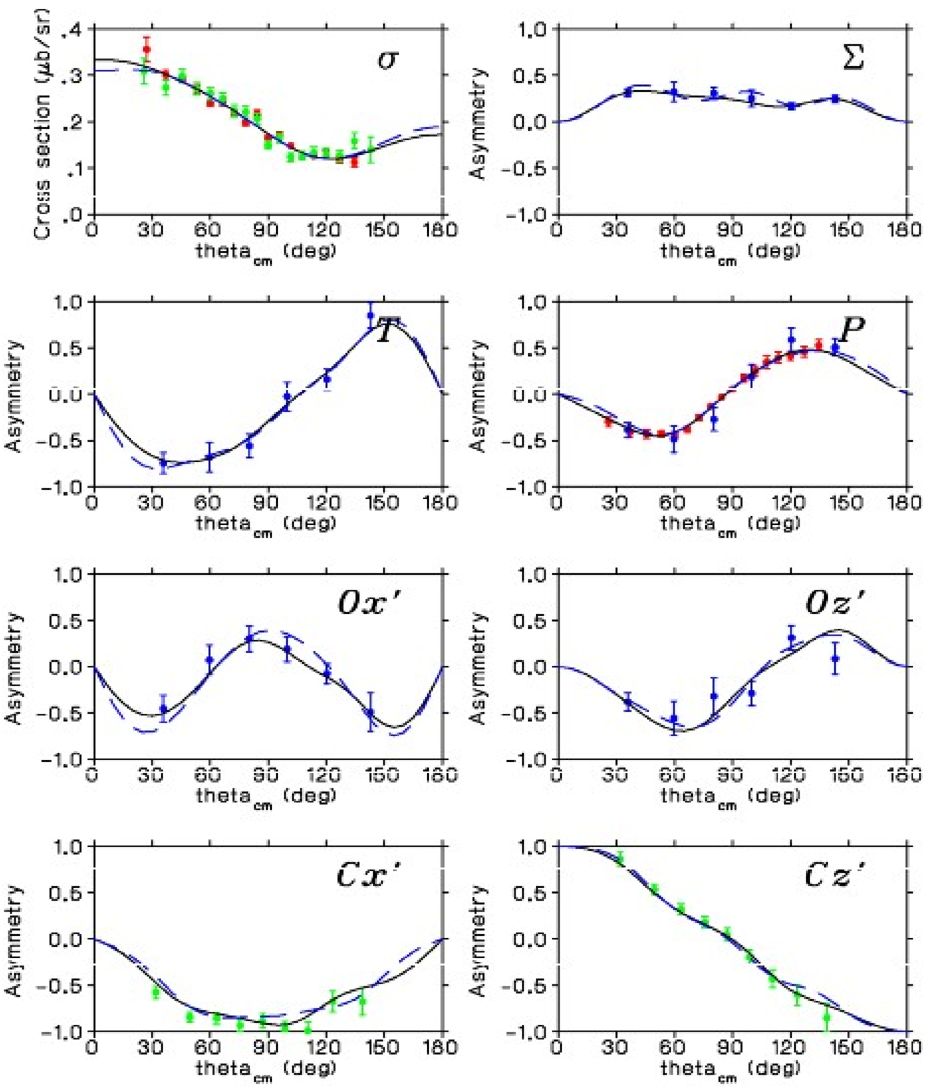
E_γ / W (MeV)	Best χ^2/pt	Largest χ^2/pt
1027 / 1676	0.49	0.54
1122 / 1728	0.59	0.62
1222 / 1781	0.52	0.62
1321 / 1833	0.74	0.92
1421 / 1883	0.97	1.15

Best χ^2 Worst χ^2

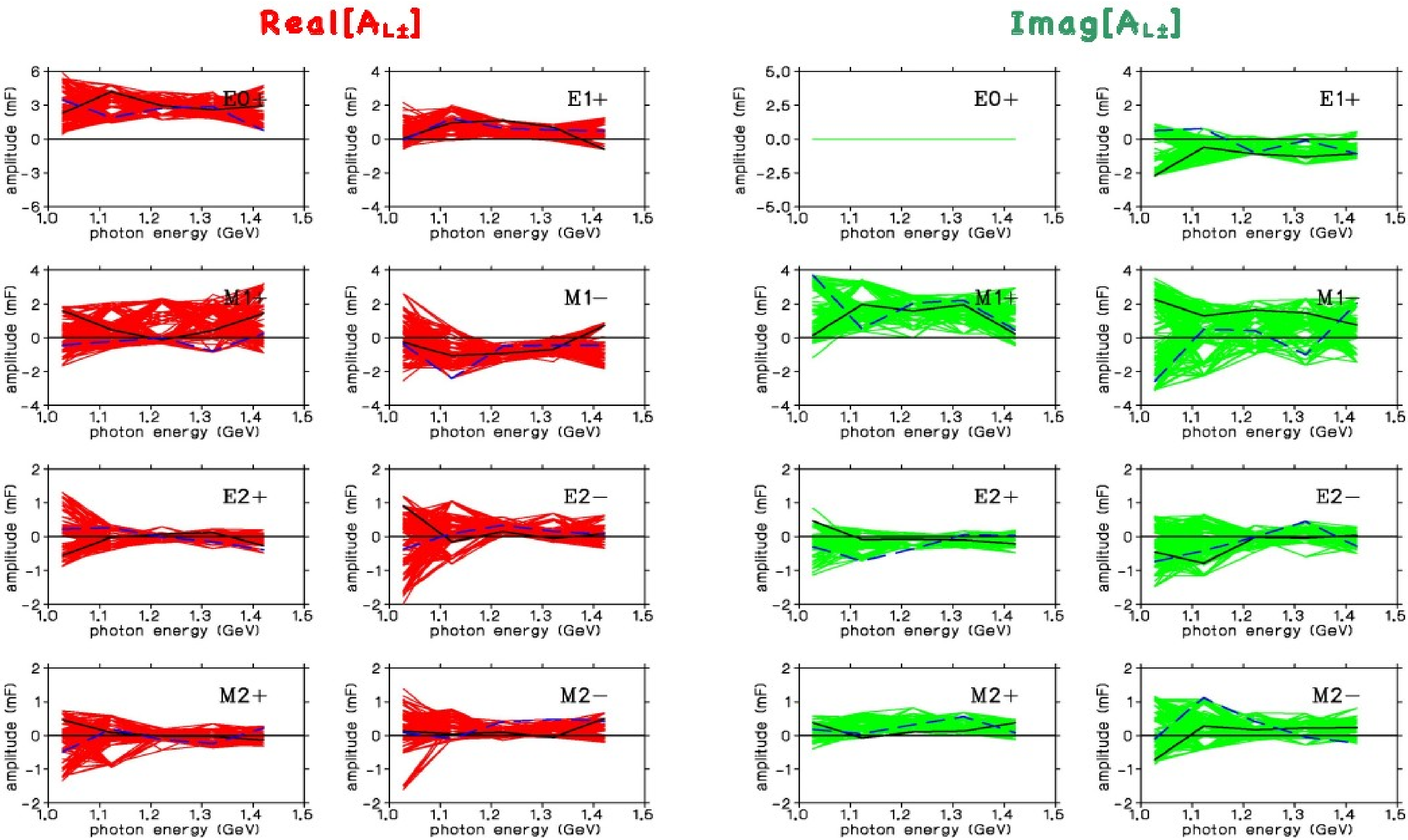
E_γ = 1122 (W = 1728) MeV



E_γ = 1421 (W = 1883) MeV



$K^+\Lambda$ Multipoles



The χ^2 surface – valley or mine-field ?

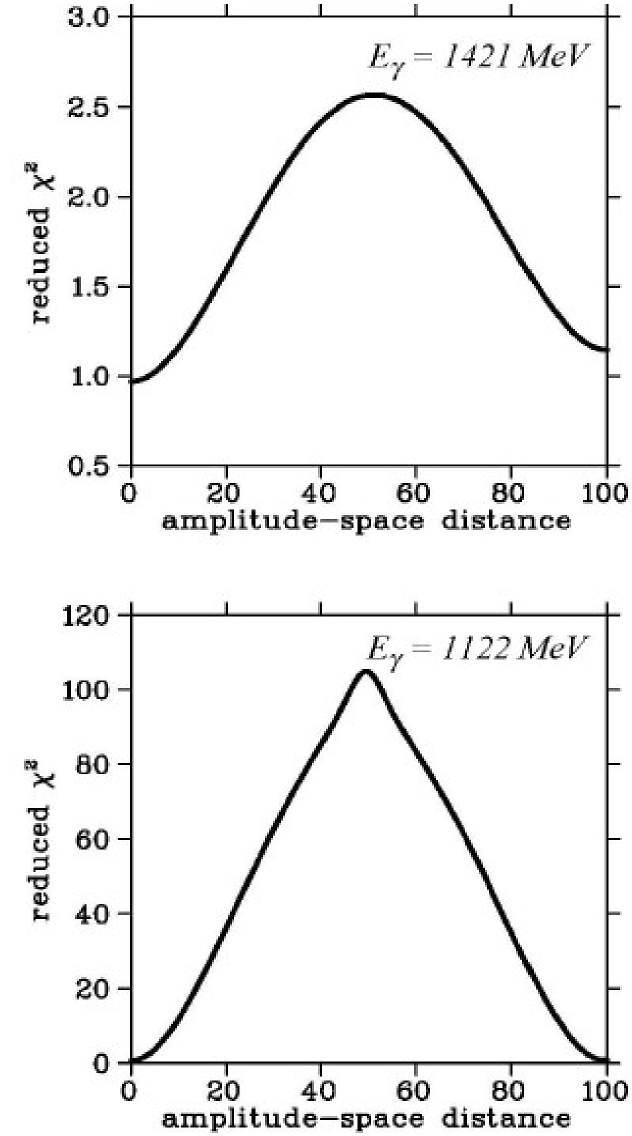
- construct a hybrid amplitude from any two solutions; track χ^2 between them.

$$A_h(x) \equiv A_1\left(1-\frac{x}{100}\right)+A_2\left(\frac{x}{100}\right),$$

$$x \in [0,\ 100]$$

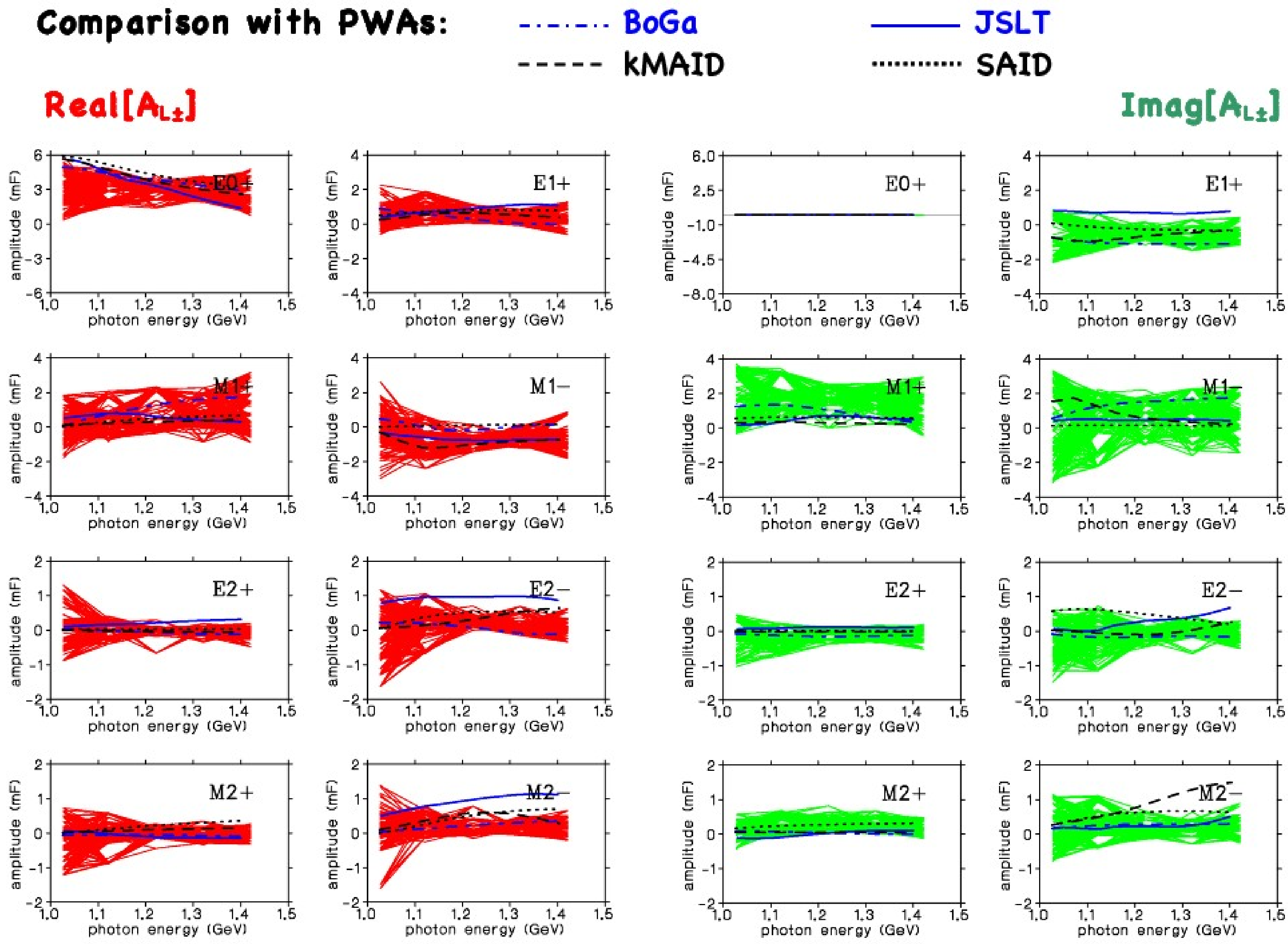
- there is always a peak between any two solutions !

$\Rightarrow$ the solution bands are clusters of many degenerate local minima



χ^2 between Best and Worst solutions

Compare to other PWAs by rotating multipoles ⇨ phase of $E_{0+} = 0$



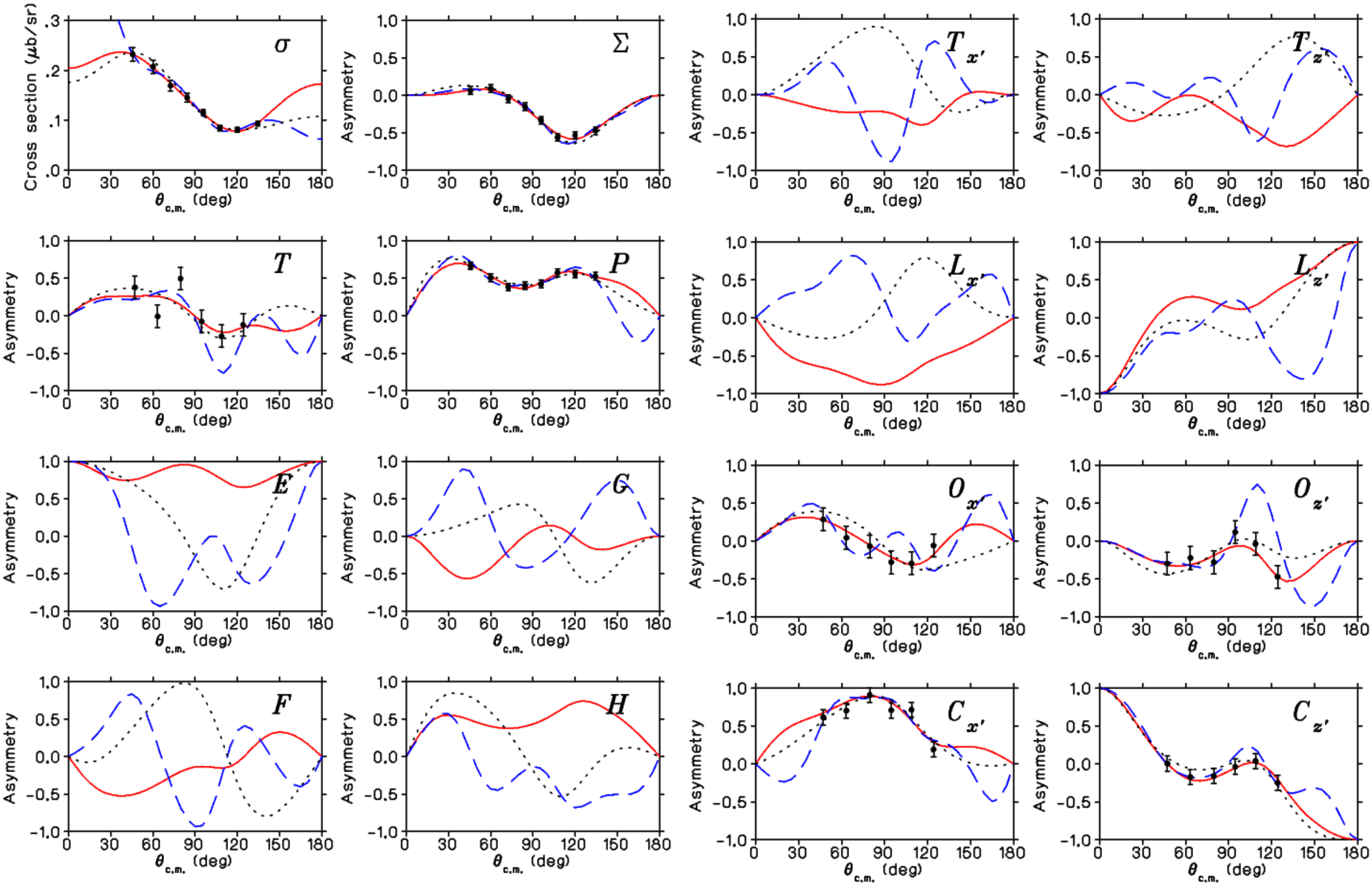
Investigate fitting "complete experiments" with mock data sets

- Generate mock data using BoGa multipoles, at energies and angles of CLAS data sets
- Distribute mock data around multipole prediction with Gaussians with a expected level of uncertainty of CLAS data
- Fit mock data varying multipoles L=0,3.

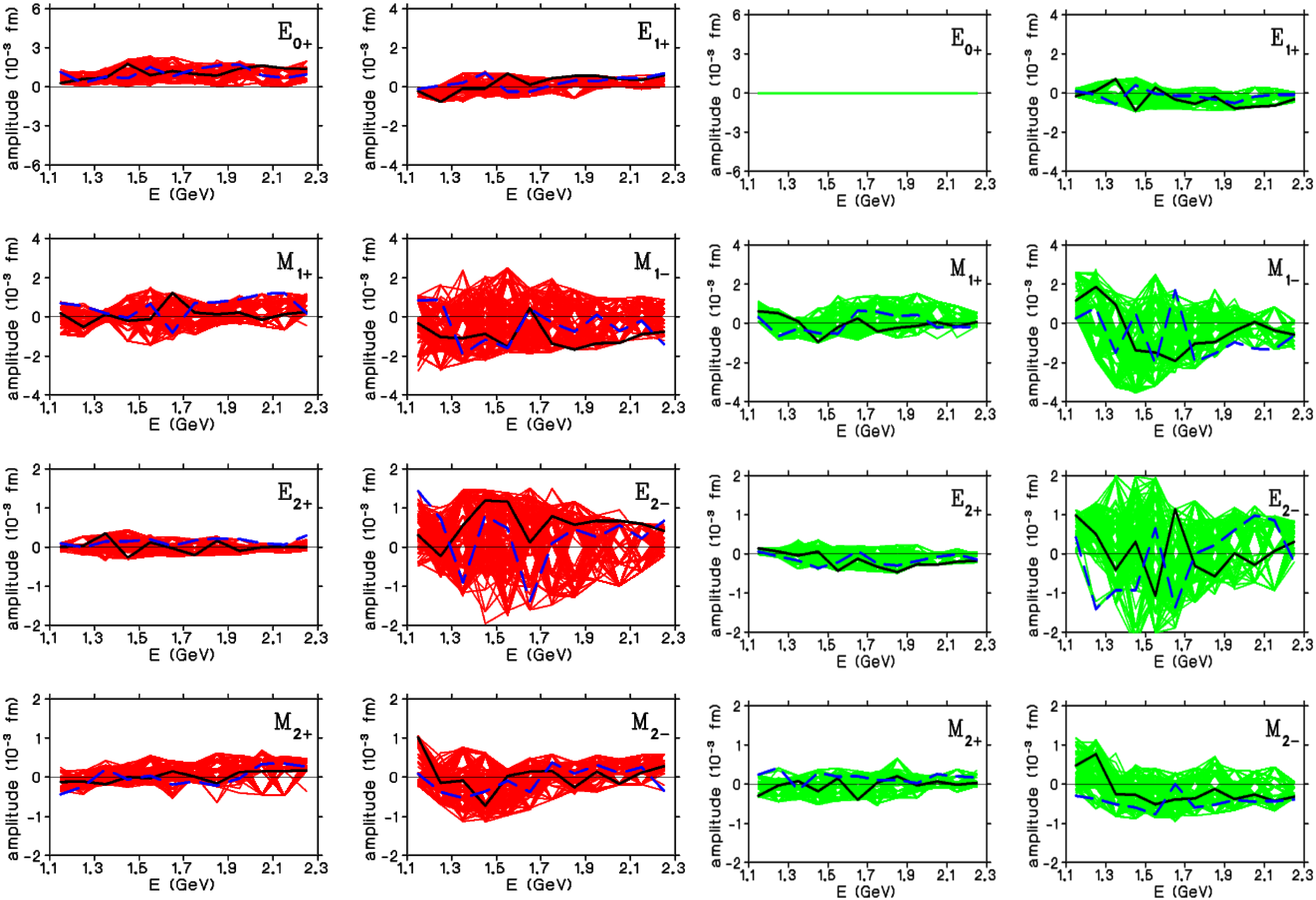
  Sample multipole space followed by gradient minimization using Minuit
- Constrain phase of $E_{0+} = 0$

J. Phys. G: Nucl. Part. Phys. **38** (2011) 053001

Partial mock data set at Eγ = 1450 MeV

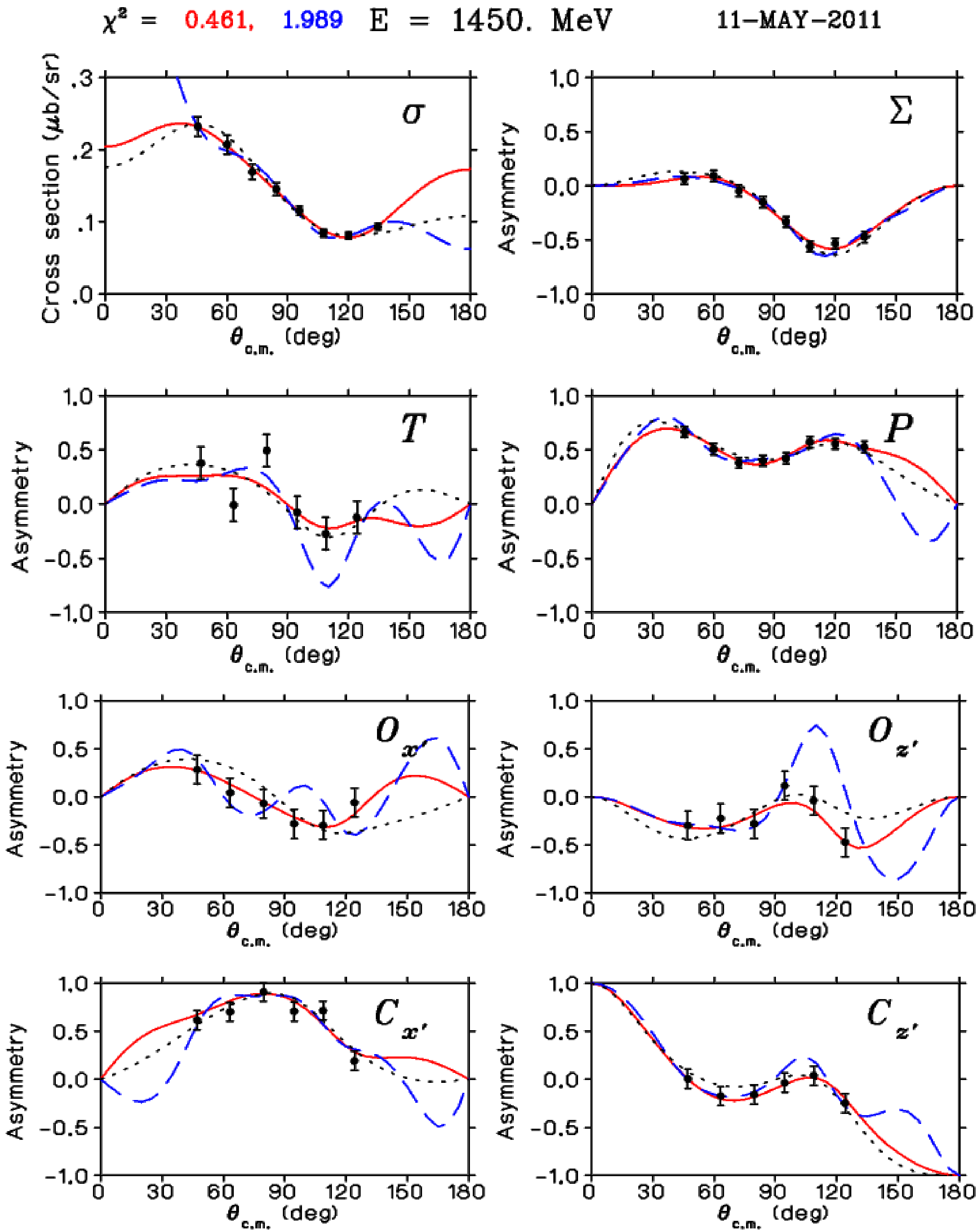


$Re[A_{l\pm}]$ $K^+ \Lambda$ $Im[A_{l\pm}]$

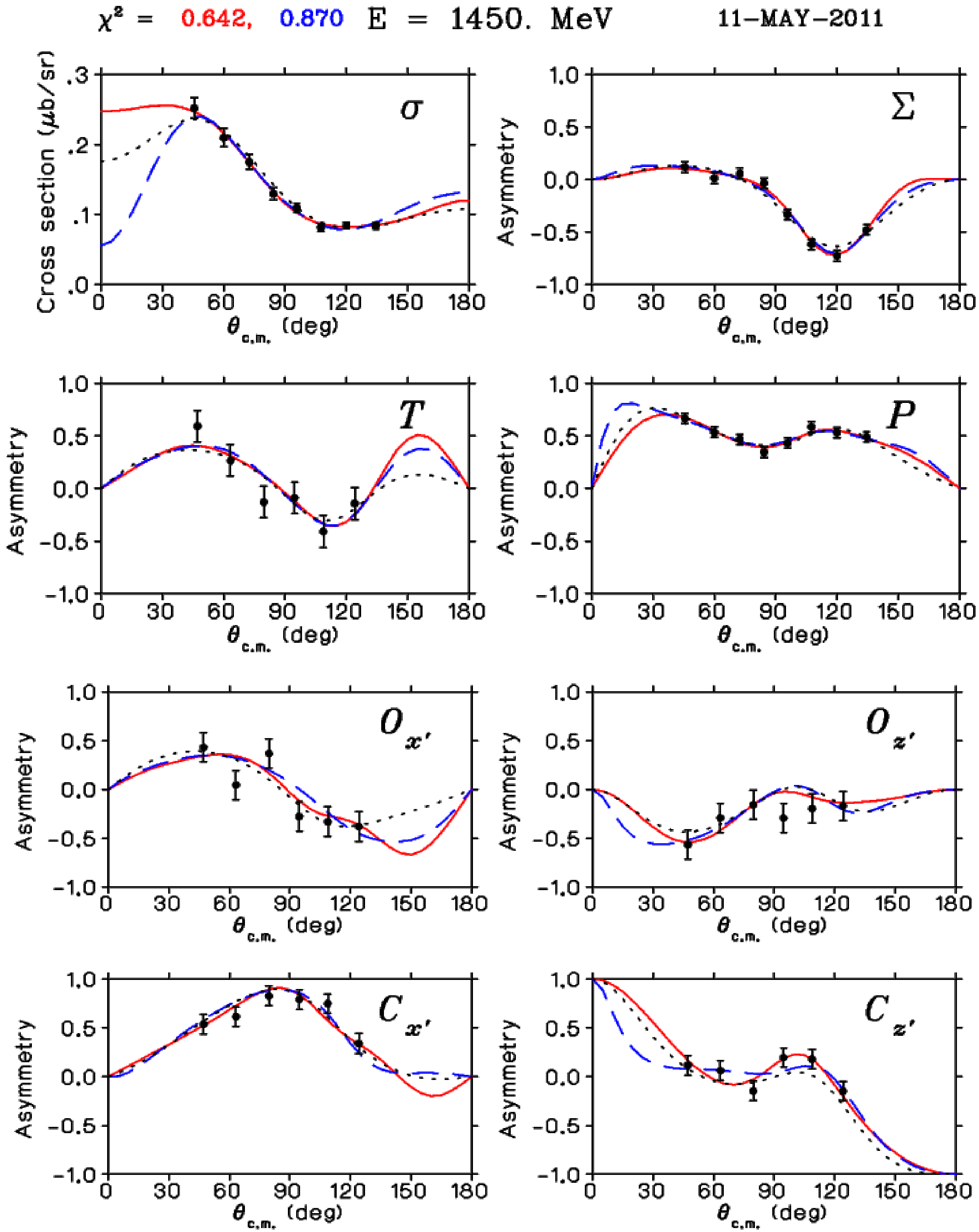


Investigate role of random distribution of a individual mock data set on fitting procedure.

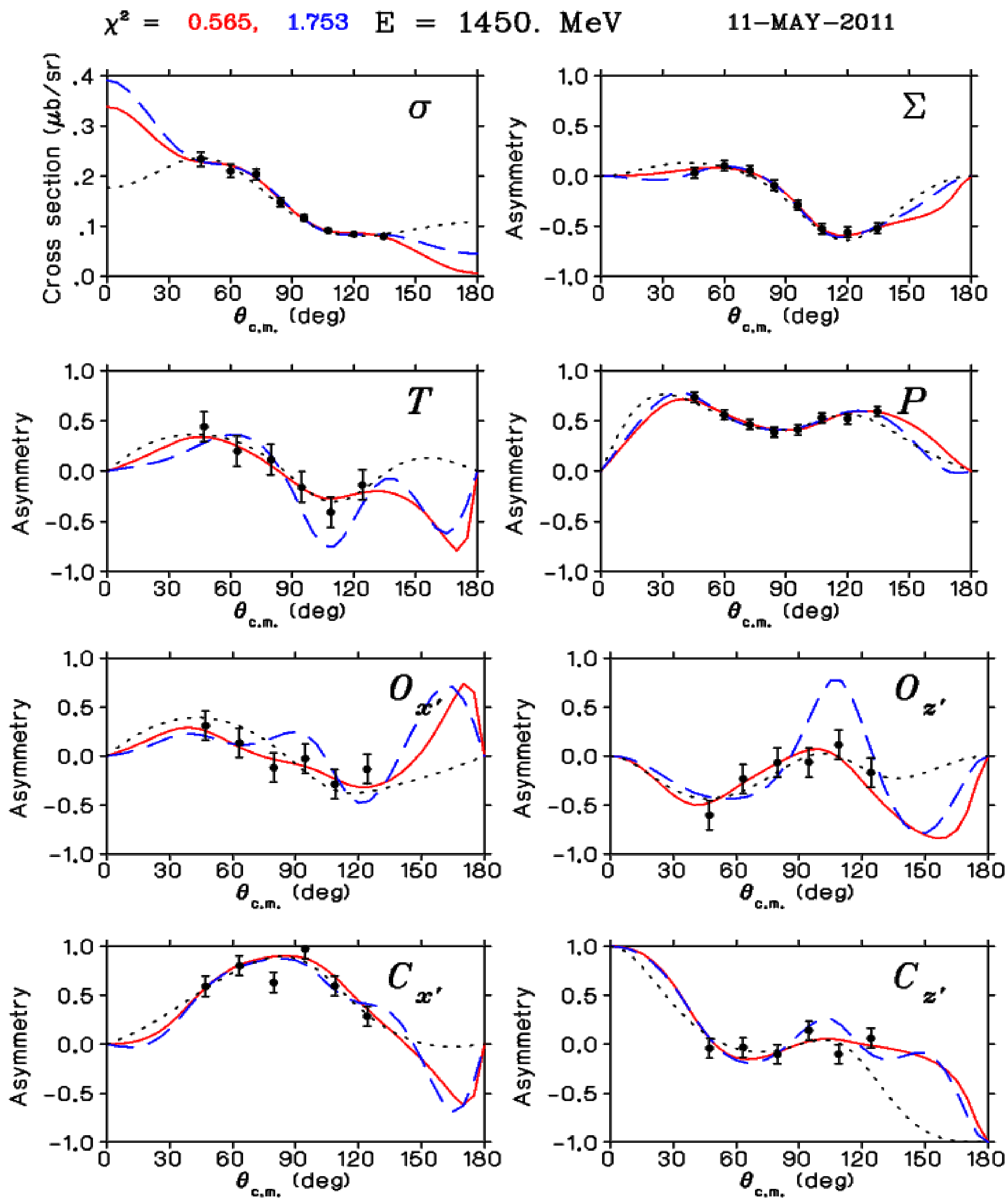
Generate various mock data sets and fit. Plot distribution of fitted observables and χ^2 distribution.



χ² = 0.461, 1.989 E = 1450. MeV 11-MAY-2011
Cross section (μb/sr)
σ
Σ
T
P
O x'
O z'
C x'
C z'
Asymmetry
θ c.m. (deg)

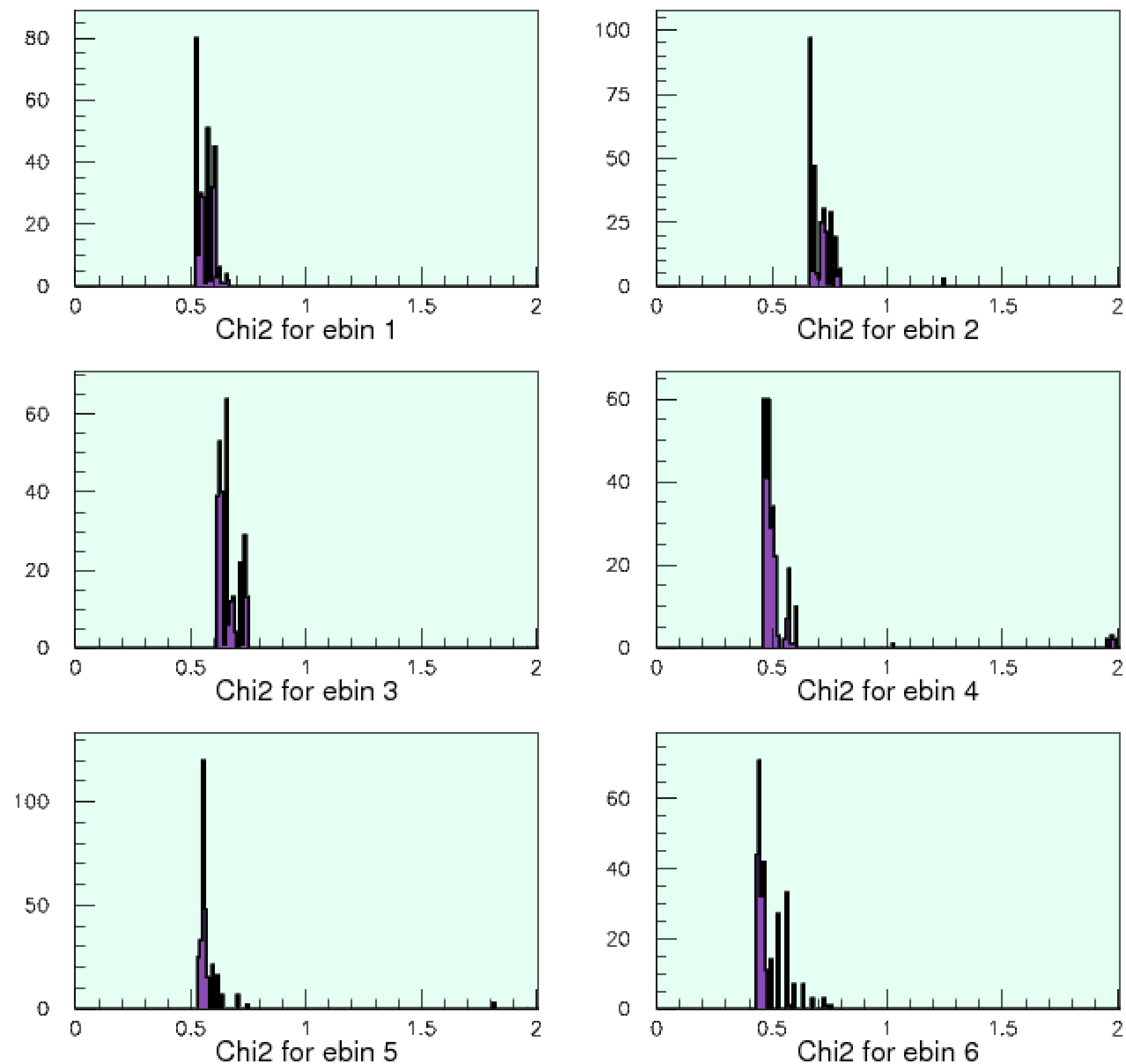


χ^2 = 0.642, 0.870 E = 1450. MeV
11-MAY-2011
Cross section (μb/sr)
σ
Σ
T
P
$O_{x'}$
$O_{z'}$
$C_{x'}$
$C_{z'}$
Asymmetry
$\theta_{c.m.}$ (deg)



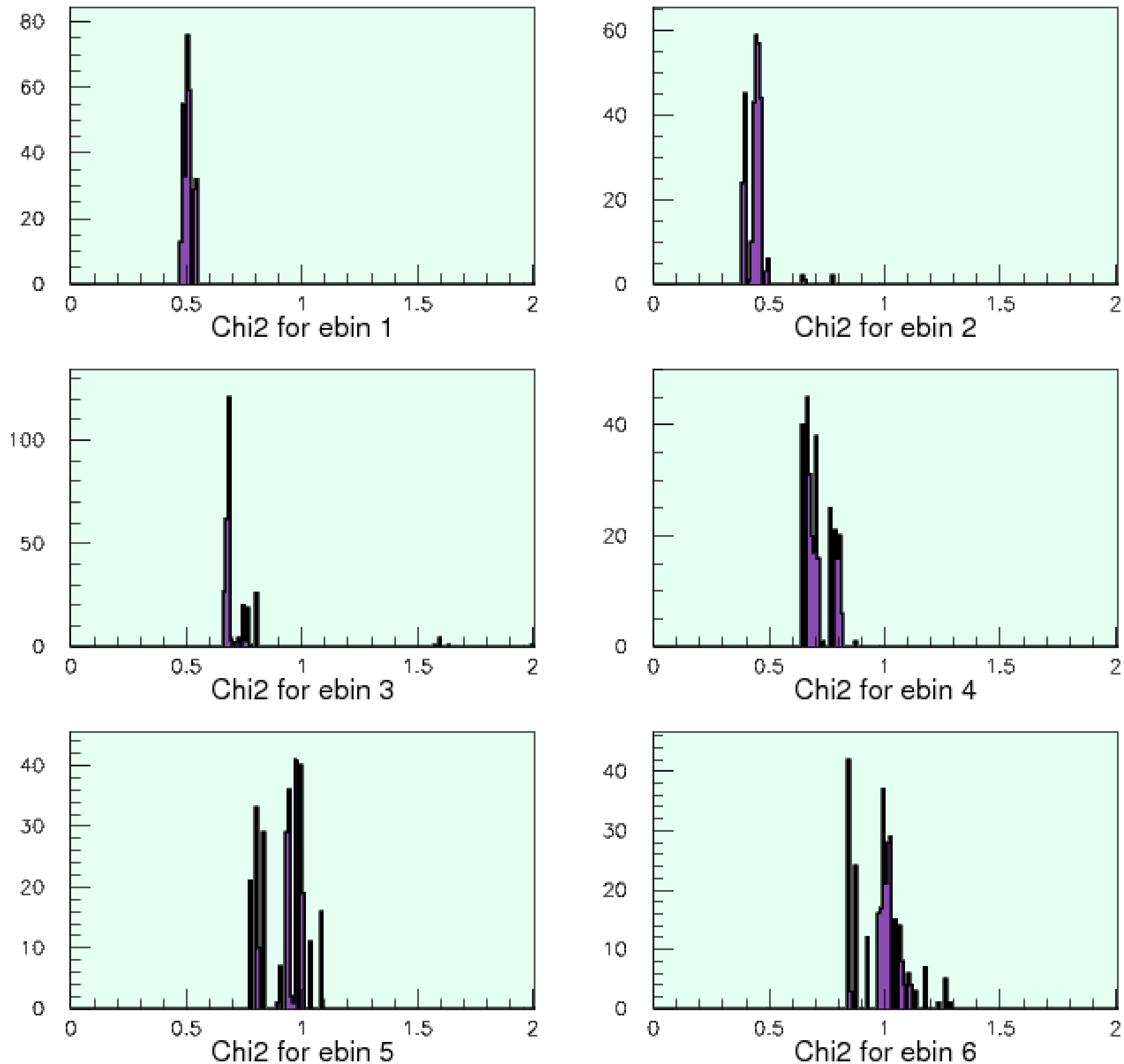
χ² = 0.565, 1.753 E = 1450. MeV 11−MAY−2011
Cross section (μb/sr)
Asymmetry
σ
Σ
T
P
O x'
O z'
C x'
C z'
θ c.m. (deg)

Chi-2 plots



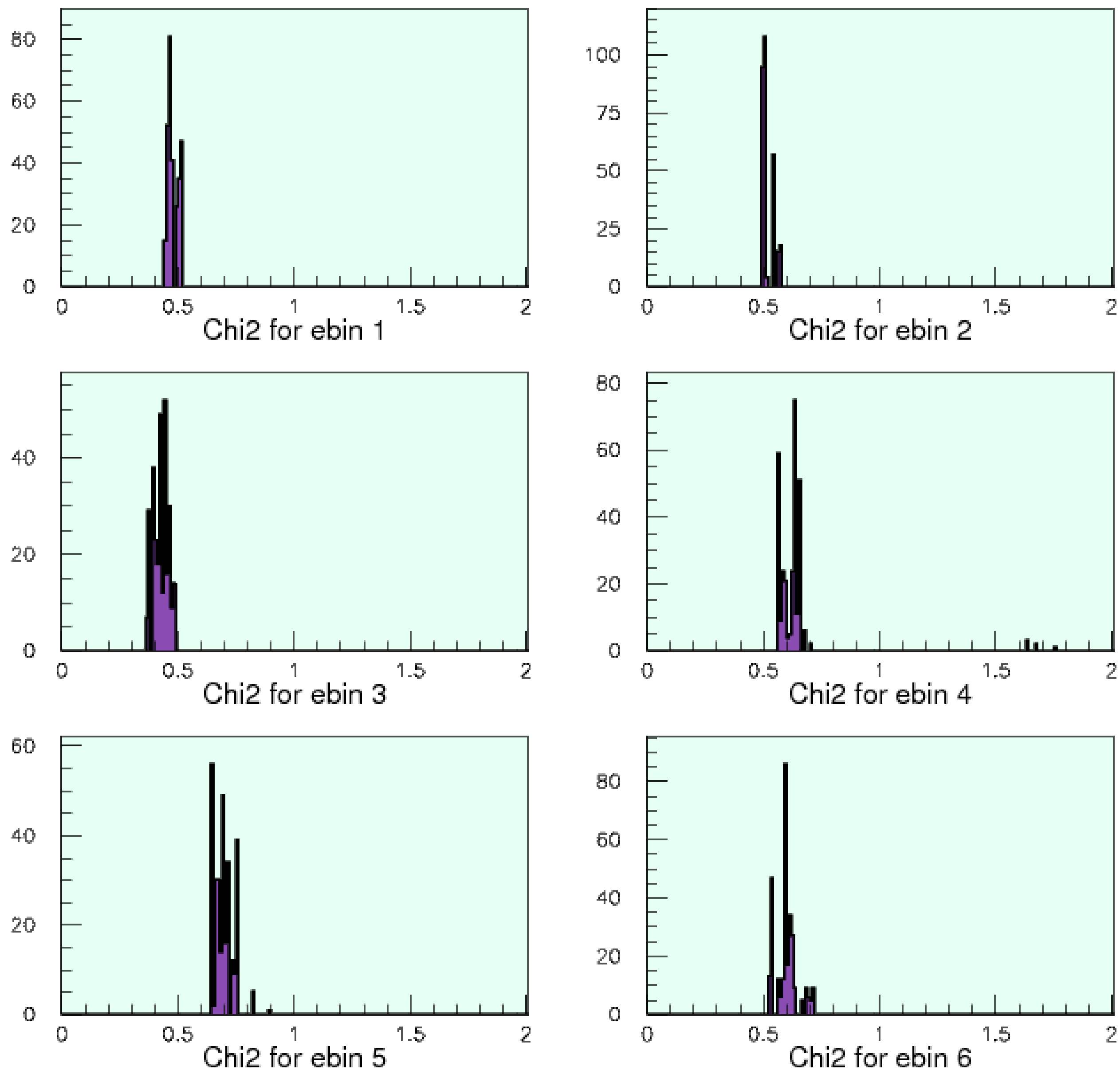
80
60
40
20
0
0
0.5
1
1.5
2
Chi2 for ebin 1
100
75
50
25
0
0
0.5
1
1.5
2
Chi2 for ebin 2
60
40
20
0
0
0.5
1
1.5
2
Chi2 for ebin 3
60
40
20
0
0
0.5
1
1.5
2
Chi2 for ebin 4
100
50
0
0
0.5
1
1.5
2
Chi2 for ebin 5
60
40
20
0
0
0.5
1
1.5
2
Chi2 for ebin 6

Chi-2 plots



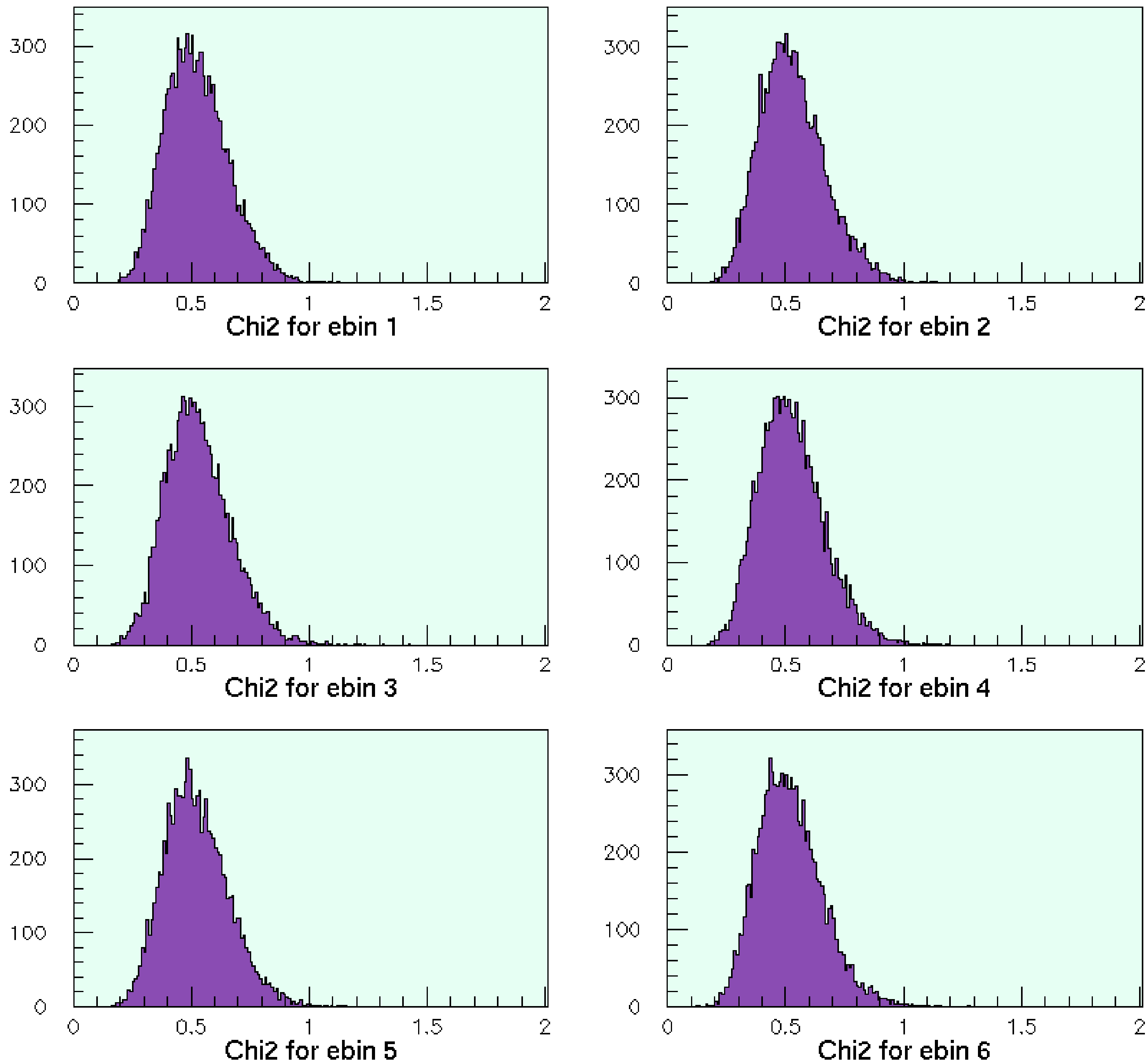
Chi2 for ebin 1
Chi2 for ebin 2
Chi2 for ebin 3
Chi2 for ebin 4
Chi2 for ebin 5
Chi2 for ebin 6

Chi-2 plots



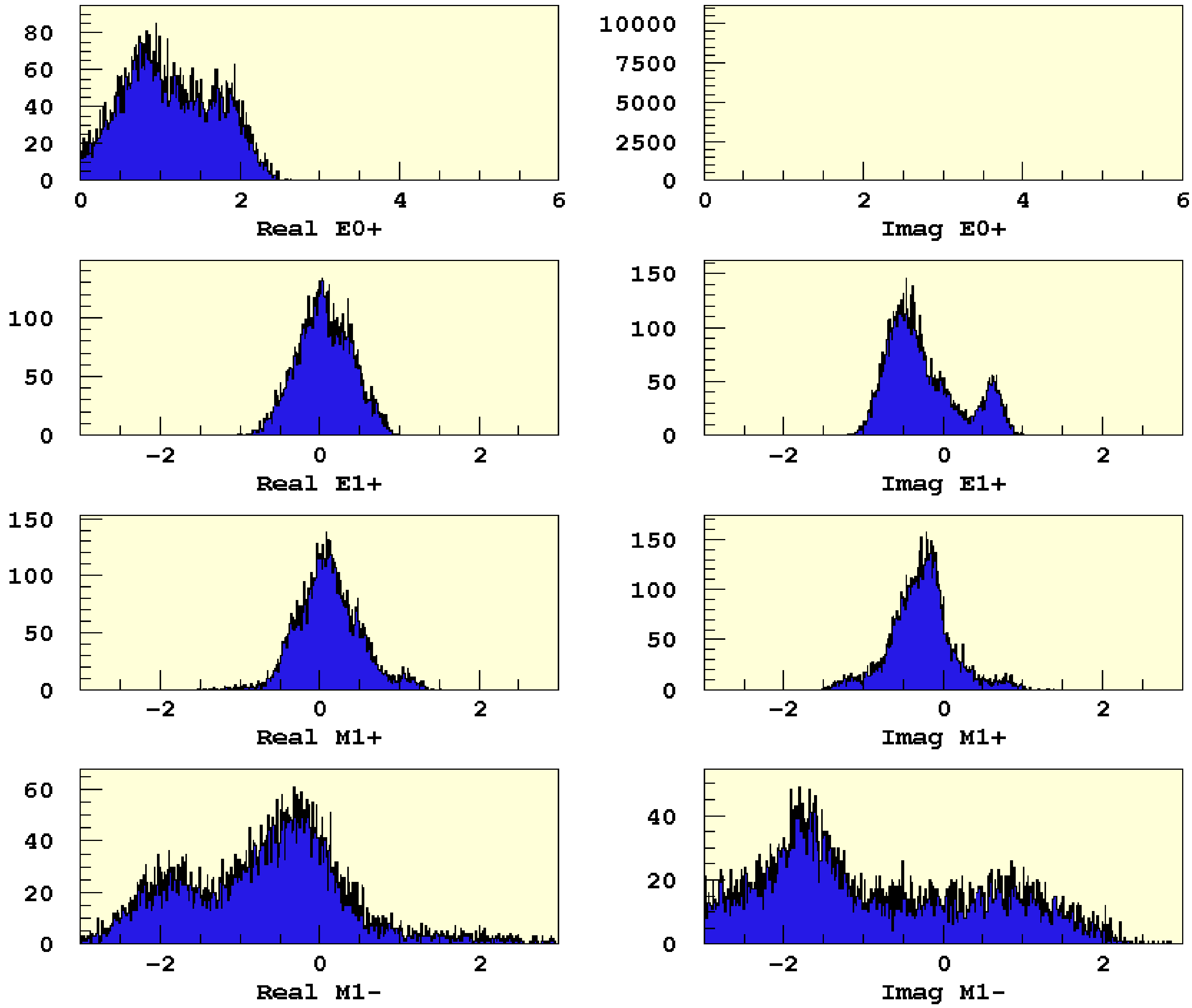
Chi2 for ebin 1
Chi2 for ebin 2
Chi2 for ebin 3
Chi2 for ebin 4
Chi2 for ebin 5
Chi2 for ebin 6

Chi-2 plots



300
200
100
0
0
0.5
1
1.5
2
Chi2 for ebin 1
Chi2 for ebin 2
Chi2 for ebin 3
Chi2 for ebin 4
Chi2 for ebin 5
Chi2 for ebin 6

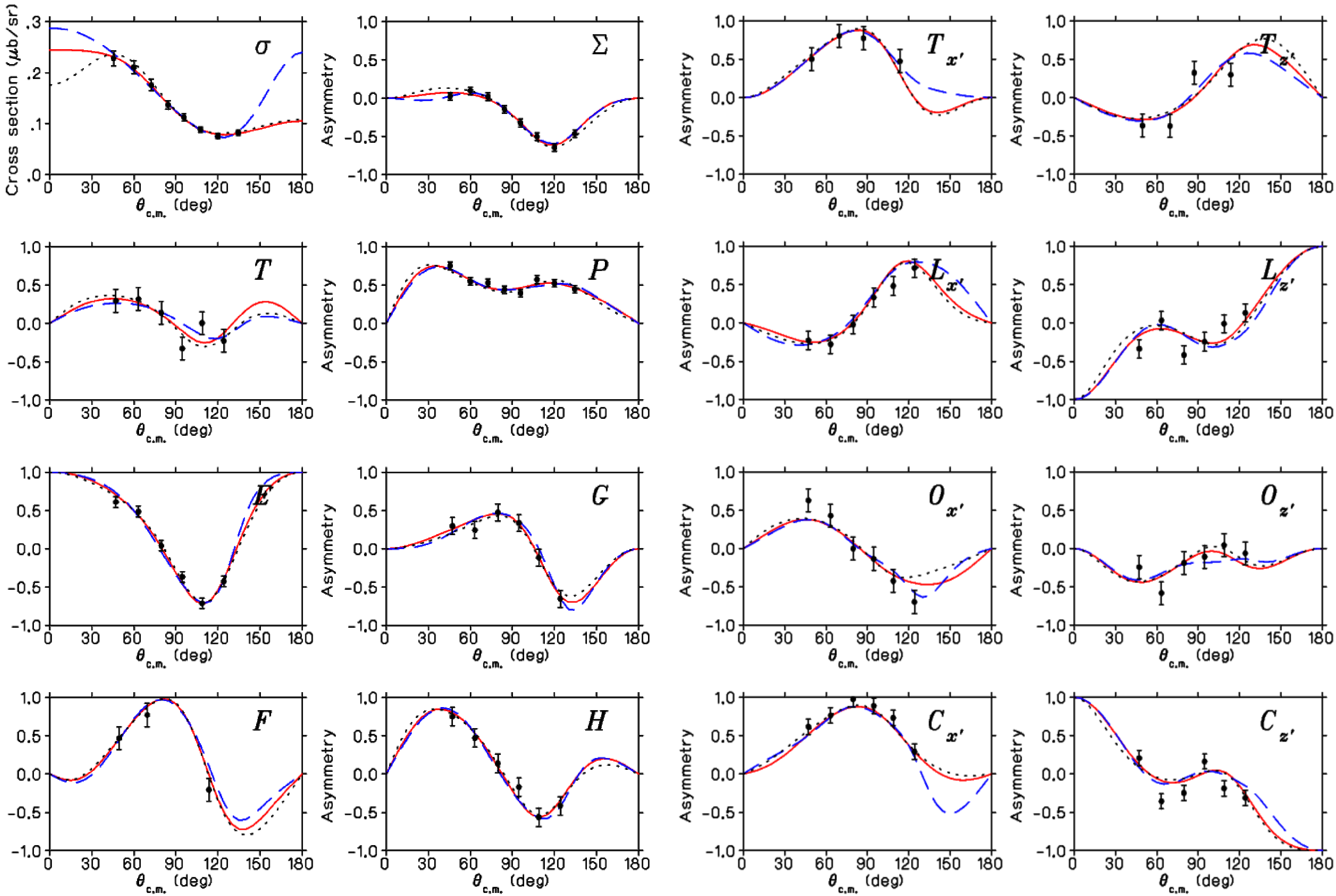
Multipoles at 1450 MeV



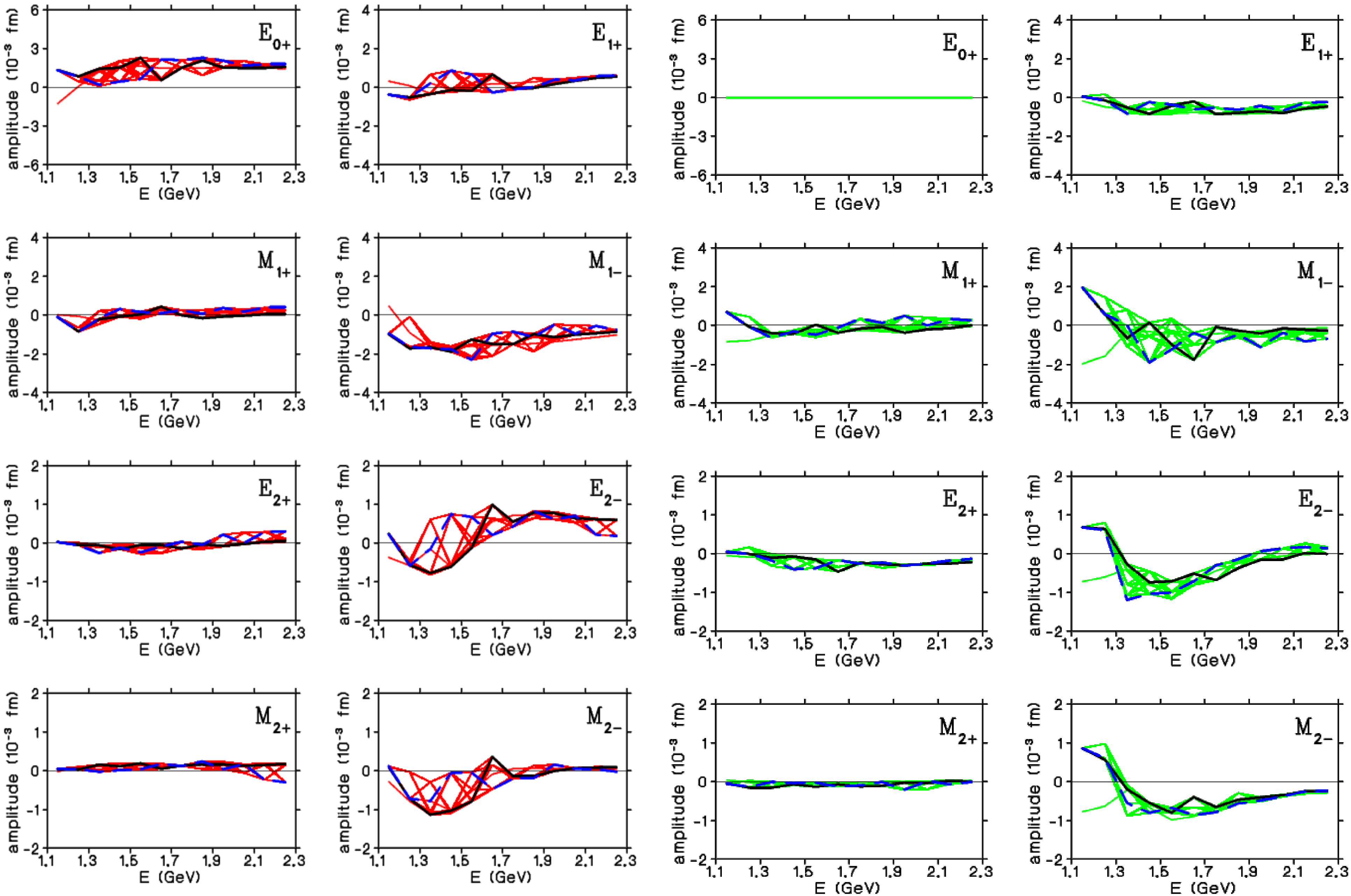
Statistics from mock fits at 1450 MeV

	Amp	Ave	Std	Minuit
Re E0+	1.664	1.139	0.573	0.290
Re E1+	-0.221	-0.059	0.336	0.203
Im E1+	-0.794	-0.253	0.458	0.202
Re M1+	-0.144	-0.115	0.379	0.224
Im M1+	-0.421	0.253	0.405	0.205
Re M1-	-1.694	-0.679	1.051	0.585
Re M1-	-0.173	-1.164	1.525	0.616

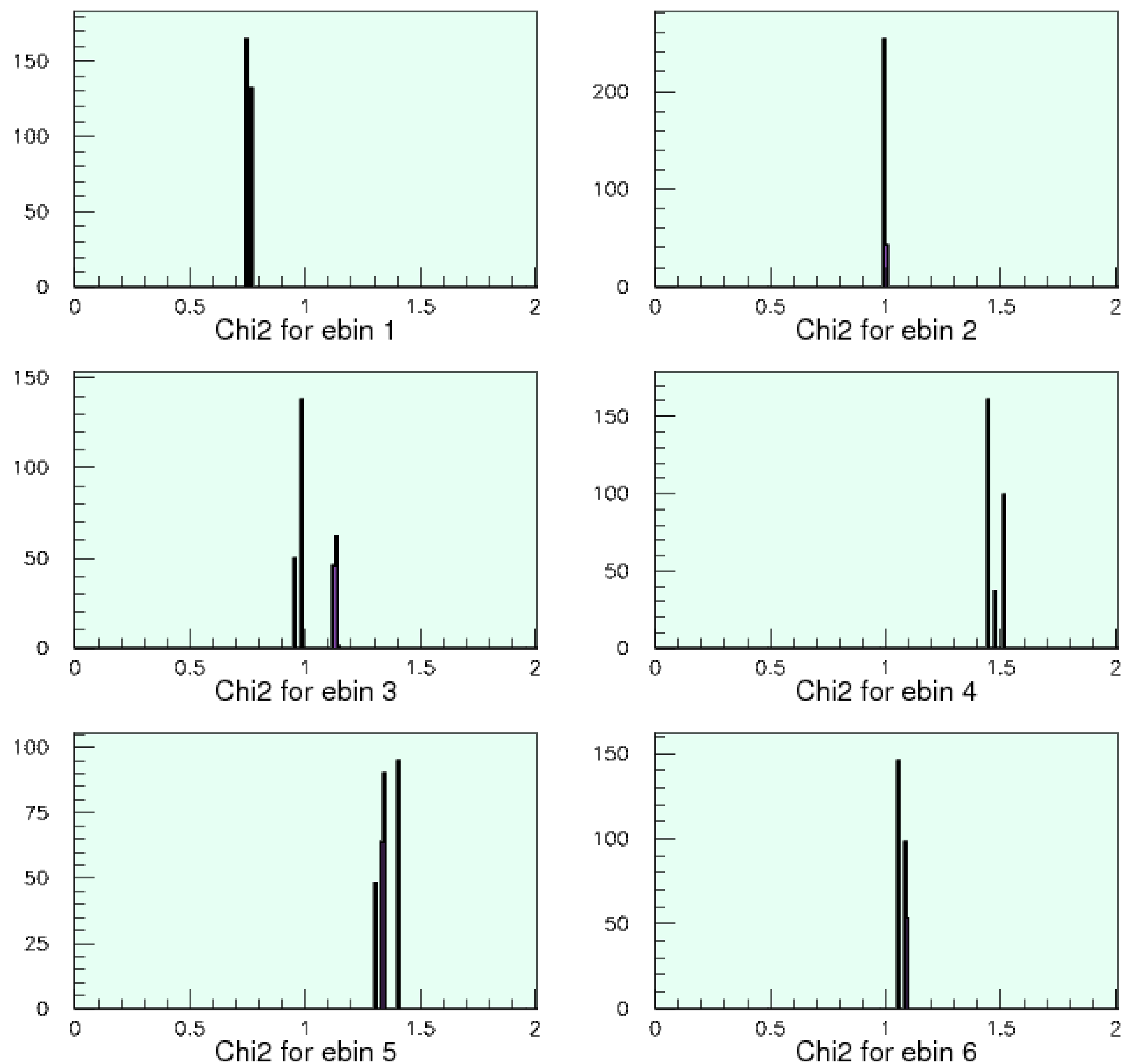
Full Mock set at E_γ=1450 MeV



$Re[A_{l\pm}]$ $K^+ \Lambda$ $Im[A_{l\pm}]$

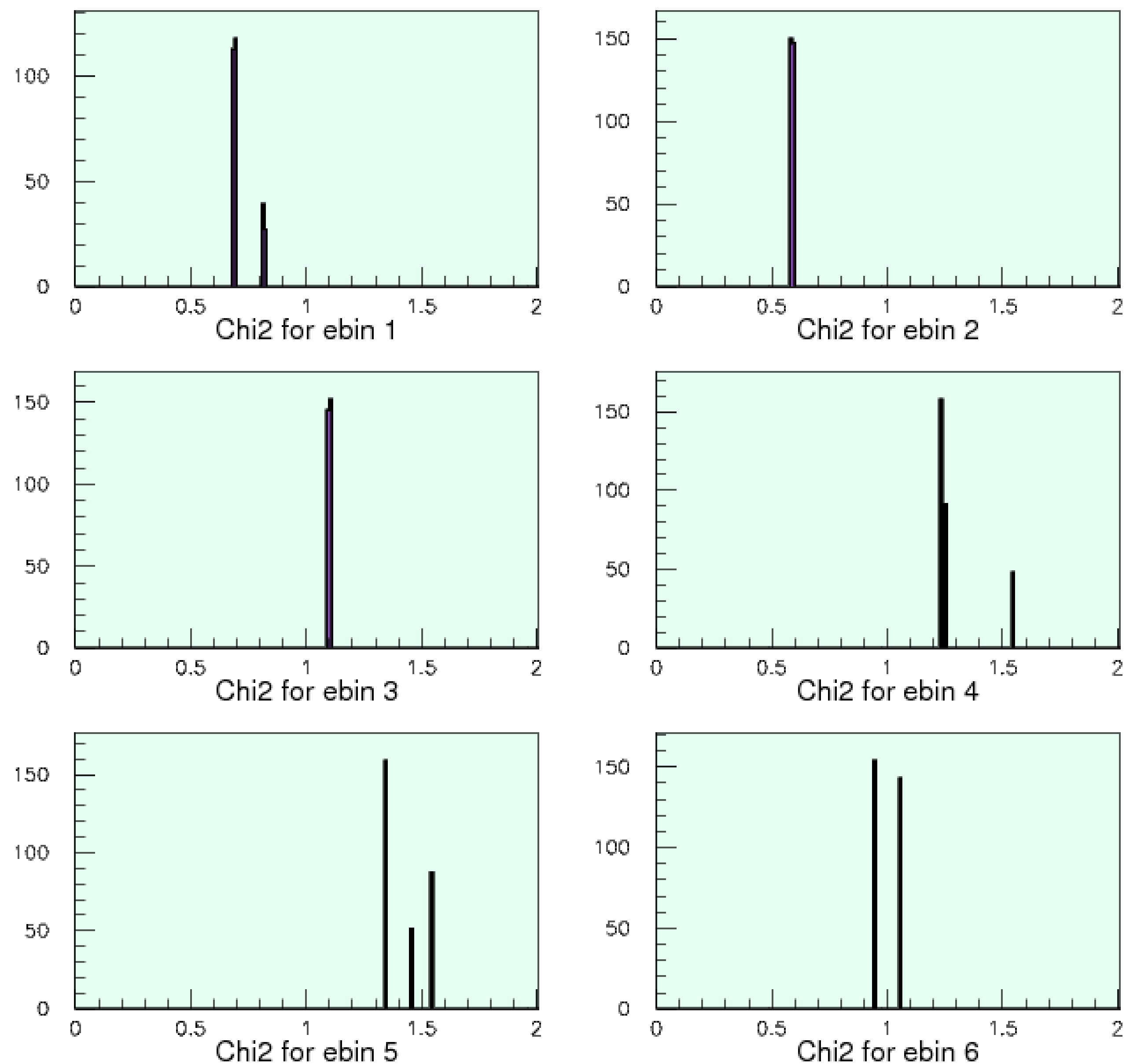


Chi-2 plots



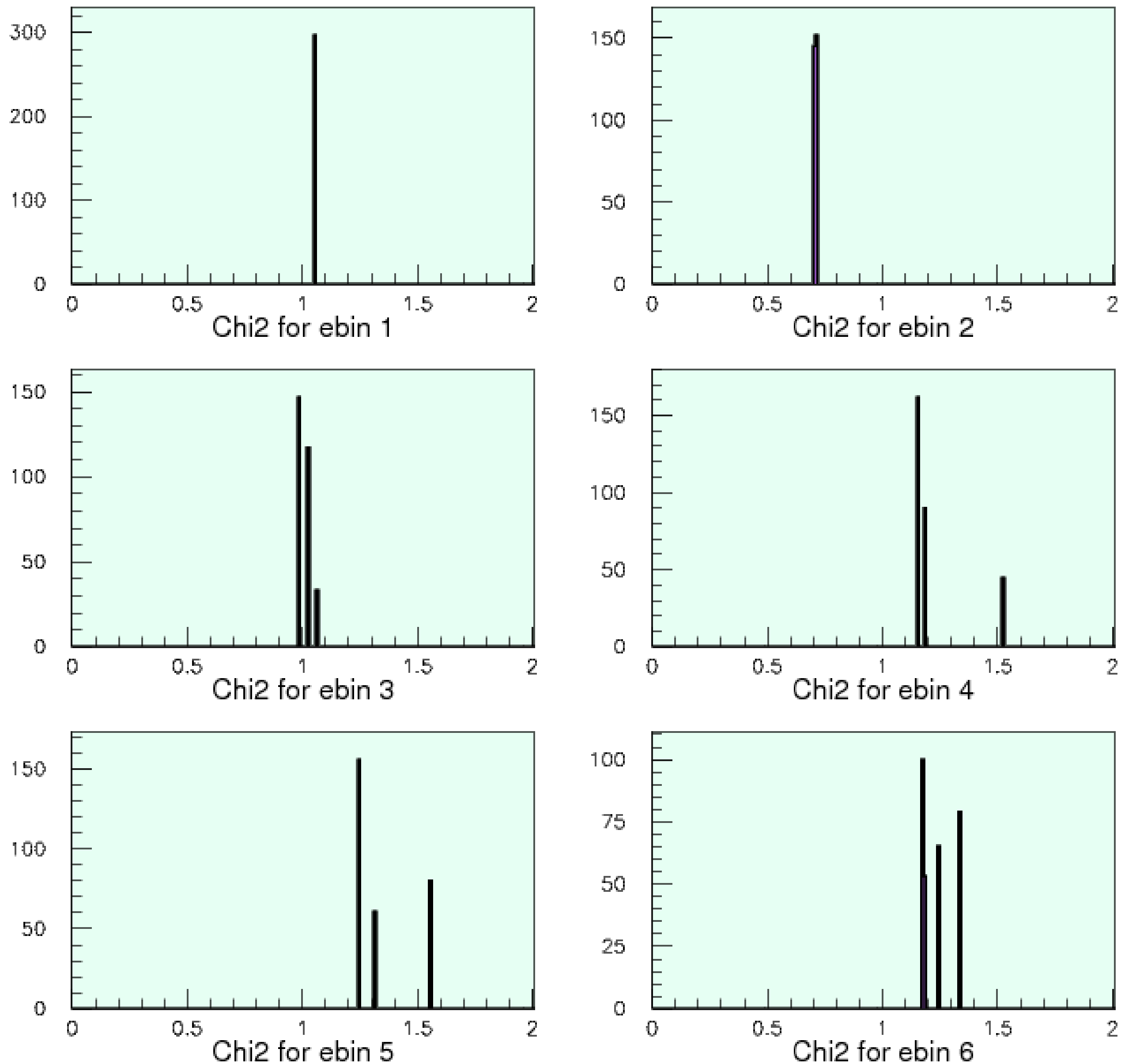
150
100
50
0
0
0.5
1
1.5
2
Chi2 for ebin 1
200
100
0
0
0.5
1
1.5
2
Chi2 for ebin 2
150
100
50
0
0
0.5
1
1.5
2
Chi2 for ebin 3
150
100
50
0
0
0.5
1
1.5
2
Chi2 for ebin 4
100
75
50
25
0
0
0.5
1
1.5
2
Chi2 for ebin 5
150
100
50
0
0
0.5
1
1.5
2
Chi2 for ebin 6

Chi-2 plots



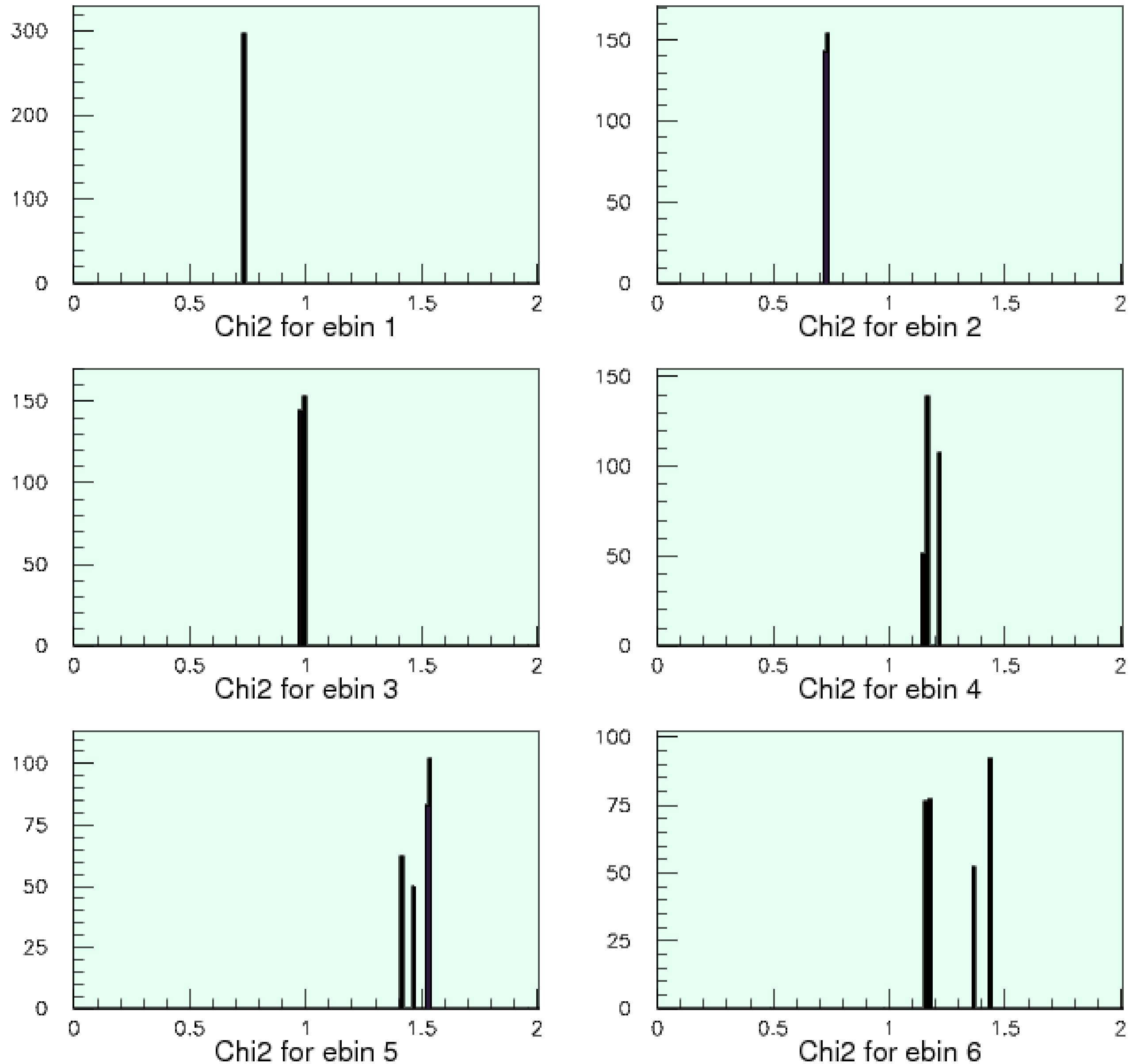
Chi2 for ebin 1
Chi2 for ebin 2
Chi2 for ebin 3
Chi2 for ebin 4
Chi2 for ebin 5
Chi2 for ebin 6
0
50
100
150
0.5
1
1.5
2

Chi-2 plots



Chi2 for ebin 1
Chi2 for ebin 2
Chi2 for ebin 3
Chi2 for ebin 4
Chi2 for ebin 5
Chi2 for ebin 6

Chi-2 plots



300
200
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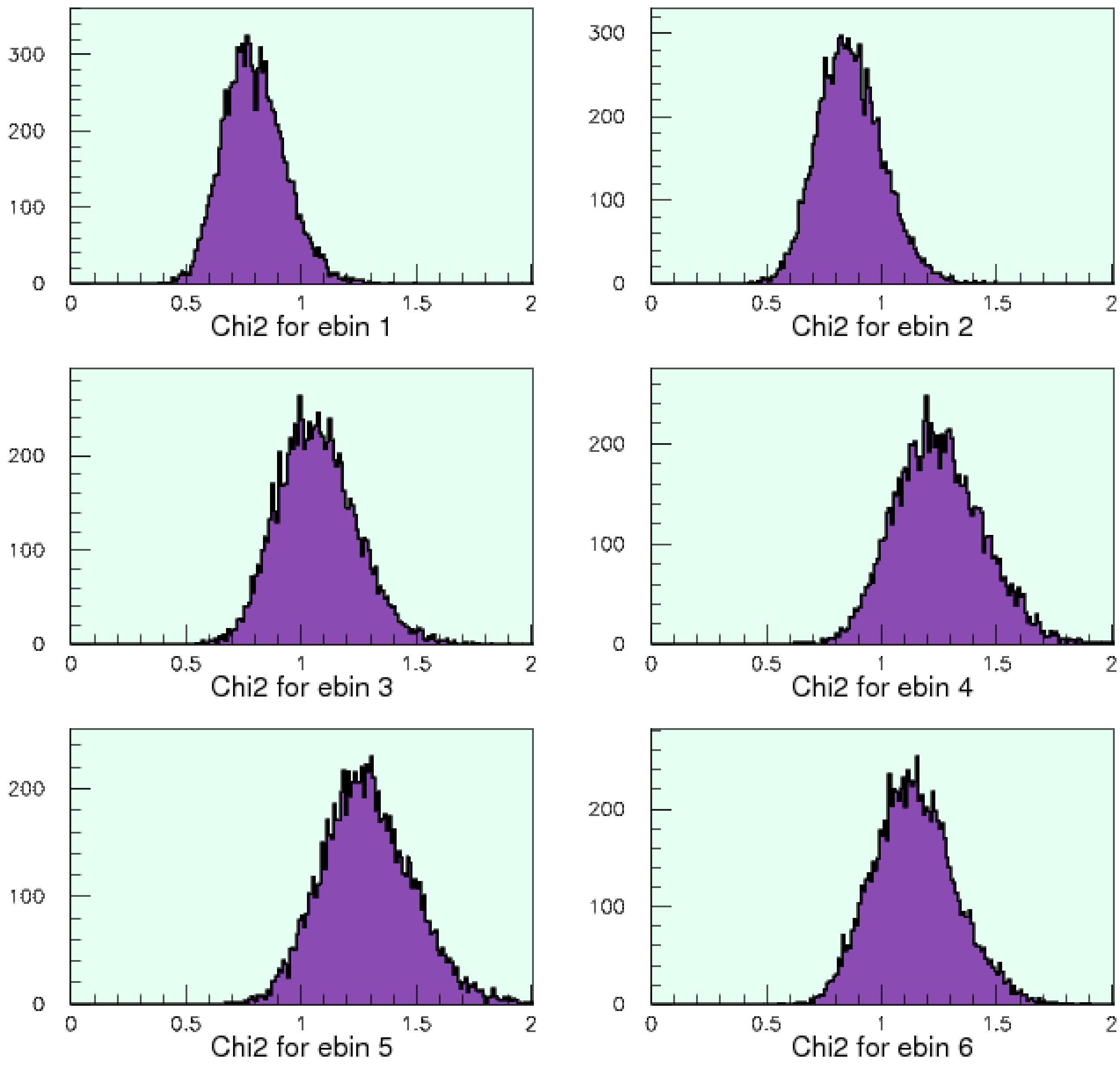
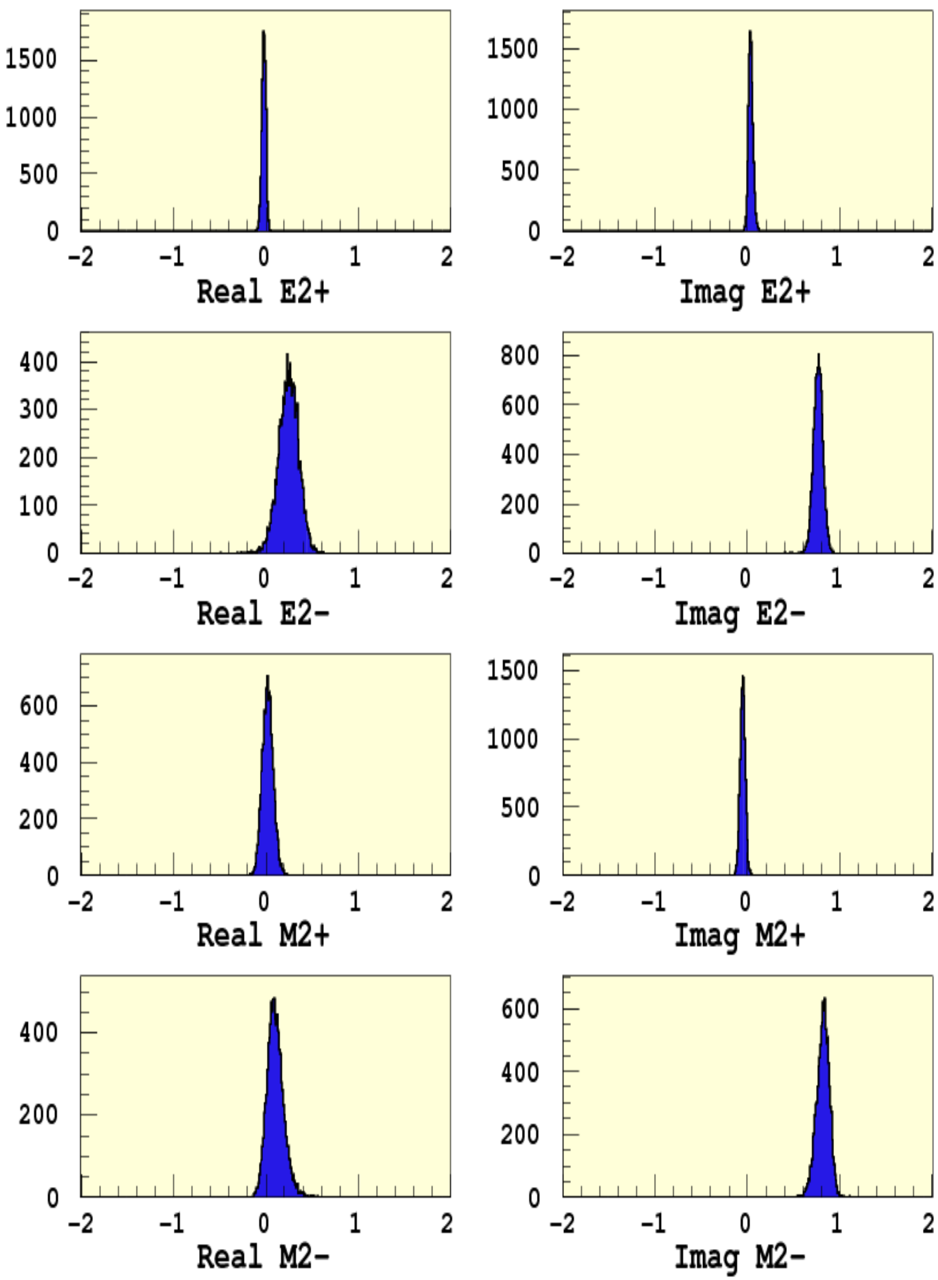
0
0.5
1
1.5
2
Chi2 for ebin 1
150
100
50
0
0
0.5
1
1.5
2
Chi2 for ebin 2
150
100
50
0
0
0.5
1
1.5
2
Chi2 for ebin 3
150
100
50
0
0
0.5
1
1.5
2
Chi2 for ebin 4
100
75
50
25
0
0
0.5
1
1.5
2
Chi2 for ebin 5
100
75
50
25
0
0
0.5
1
1.5
2
Chi2 for ebin 6

Chi-2 plots

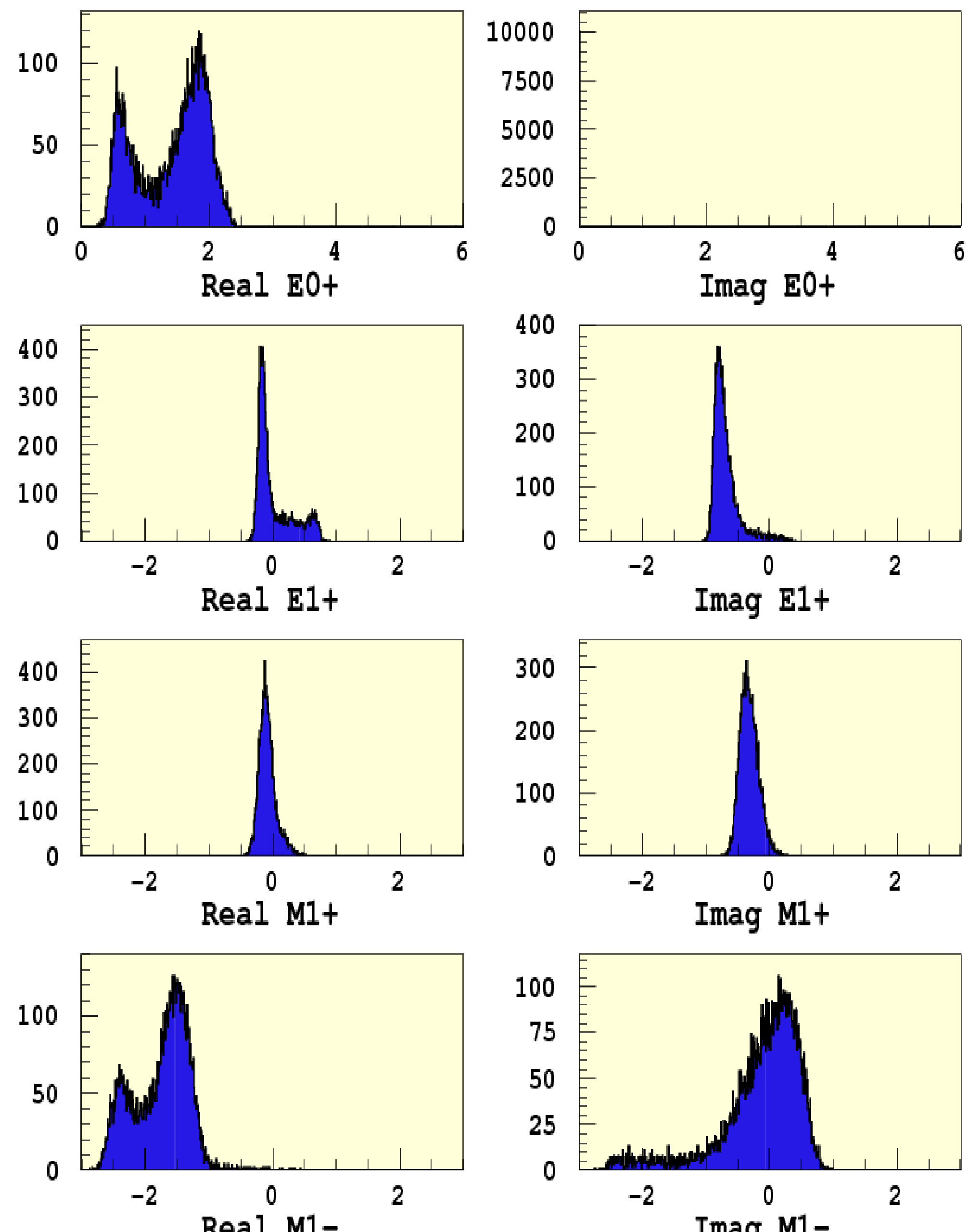


300
200
100
0
0
0.5
1
1.5
2
Chi2 for ebin 1
300
200
100
0
0
0.5
1
1.5
2
Chi2 for ebin 2
200
100
0
0
0.5
1
1.5
2
Chi2 for ebin 3
200
100
0
0
0.5
1
1.5
2
Chi2 for ebin 4
200
100
0
0
0.5
1
1.5
2
Chi2 for ebin 5
200
100
0
0
0.5
1
1.5
2
Chi2 for ebin 6

E_γ = 1150 MeV



E_γ = 1550 MeV



Statistics from mock fits at 1550 MeV

	Amp	Ave	Std	Minuit
Re E0+	1.714	1.412	0.547	0.205
Re E1+	-0.146	0.025	0.287	0.127
Im E1+	-0.807	-0.695	0.207	0.149
Re M1+	-0.082	-0.083	0.132	0.109
Im M1+	-0.344	-0.323	0.143	0.106
Re M1-	-1.665	-1.755	0.435	0.301
Im M1-	0.018	-0.136	0.638	0.475

Summary

- With realistic experimental uncertainties as expected from ongoing experiments, fits to multipoles will always have multiple minima that are experimentally indistinguishable.
- The position of these minima, and the values of the assoc. multipoles, depend on the statistical distribution in the data set.
- Errors reported by standard minimization packages such as MINUIT reflect the curvature of the χ^2 space around one local minimum, and thus generally underestimate the true uncertainty in the presence of multiple minima.
- In fitting the data from a complete experiment mock data with the same errors can be used to evaluate the multipole uncertainties associated with the variations in the statistical distributions of data points.
- In fits to a “complete set” of all 16 observables, with the uncertainties expected from ongoing experiments, the minima are fairly well clustered, with the result that an extracted amplitude is expected to be fairly well determined.